MAXIMAL $L^p$ ERROR ANALYSIS OF FEMS FOR NONLINEAR PARABOLIC EQUATIONS WITH NONSmooth COEFFICIENTS

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in a polyhedral domain in \( \mathbb{R}^N \), \( N = 2, 3 \), and its semi-discrete finite element approximation

\[
\begin{cases}
(\partial_t u_h, v_h) + \sum_{i,j=1}^{N} (\sigma_{ij}(u_h, x) \partial_j u_h, \partial_i v_h) = (g(u_h, \nabla u_h, x), v_h), \forall v_h \in S_h, \\
u_h(0) = u_h^0,
\end{cases}
\]

where \( S_h \) denotes a finite element subspace of \( H_0^1(\Omega) \) consisting of continuous piece-wise polynomials of degree \( r \geq 1 \) subject to a quasi-uniform triangulation of \( \Omega \) with a mesh size \( h \), and \( u_h^0 = I_h u^0 \) denotes the Lagrange interpolation of the initial data \( u^0 \). We only impose certain local conditions on the coefficients \( \sigma_{ij}(u, x) = \sigma_{ji}(u, x) \) and the right-hand side \( g(u, \nabla u, x) \), i.e. we assume that for \( |u| \leq M, |\eta| \leq M, x \in \Omega \) and \( t \in [0, T] \)

\[
|\sigma_{ij}| + |\partial_u \sigma_{ij}| + |\partial_x \sigma_{ij}| \leq K_M,
\]

\[
K_M^{-1} |\xi|^2 \leq \sum_{i,j=1}^{N} \sigma_{ij}(u, x) \xi_i \xi_j \leq K_M |\xi|^2,
\]

\[
|g(u, \eta, x)| + |\partial_u g(u, \eta, x)| + |\partial_x g(u, \eta, x)| + |\partial_{x_j} g(u, \eta, x)| + |\partial_{x_j \eta_j} g(u, \eta, x)| + |\partial_{x_j \eta_j} g(u, \eta, x)| \leq K_M,
\]

for some positive constant \( K_M \) which may depend on \( M \), where \( \partial_u, \partial_x \) and \( \partial_x \) denote the partial derivatives with respect to \( u, x \) and \( \eta_j \), and \( \partial_{u, \eta_j} \) and \( \partial_{x_j \eta_j} \) denote the mixed second-order partial derivatives.

The key to the optimal error estimate for the nonlinear problem (2) is more precise \( L^p \) estimates of the finite element solution, defined by

\[
\begin{cases}
(\partial_t \phi_h, v_h) + \sum_{i,j=1}^{N} (a_{ij} \partial_j \phi_h, \partial_i v_h) = (f, v_h), \forall v_h \in S_h, \\
\phi_h(0) = \phi^0_h
\end{cases}
\]

for the linear parabolic equation

\[
\begin{cases}
\partial_t \phi - \sum_{i,j=1}^{N} \partial_i (a_{ij} \partial_j \phi) = f & \text{in } \Omega \times (0, \infty), \\
\phi = 0 & \text{on } \partial \Omega \times (0, \infty), \\
\phi(\cdot, 0) = \phi^0 & \text{in } \Omega
\end{cases}
\]

where \( a_{ij} = a_{ji} \). Namely, the discrete maximal \( L^p \)-regularity

\[
\| \partial_t \phi_h \|_{L^p(0,T;L^p)} + \| A_h \phi_h \|_{L^p(0,T;L^p)} \leq C_{p,q} \| f \|_{L^p(0,T;L^p)}, \quad \text{if } \phi^0 = 0,
\]

\[
\| \partial_t \phi_h \|_{L^p(0,T;W^{-1,q})} + \| \phi_h \|_{L^p(0,T;W^{1,q})} \leq C_{p,q} \| f \|_{L^p(0,T;W^{-1,q})}, \quad \text{if } \phi^0 = 0,
\]
and the optimal-order error estimate

\[ \| P_h \phi - \phi_h \|_{L^p(0,T;L^q)} \leq C_{p,q} \| P_h \phi^0 - \phi_h^0 \|_{L^q} + C_{p,q} \| P_h \phi - R_h \phi \|_{L^p(0,T;L^q)}. \]

Here \( A_h(t) : S_h \rightarrow S_h \) is the discrete counterpart of the differential operator \( A(t)u = -\partial_j(a_{ij}(\cdot, t) \partial_i u) \), defined by

\[ (A_h w_h, v_h) := \sum_{i,j=1}^{N} (a_{ij} \partial_j w_h, \partial_i v_h), \forall w_h, v_h \in S_h, \]

\( R_h(t) \) is the Ritz projection operator onto the finite element space, defined by

\[ \sum_{i,j=1}^{N} (a_{ij} \partial_j (w - R_h w), \partial_i v_h) = 0, \forall w \in H^1_0(\Omega), v_h \in S_h, \]

and \( P_h \) is the \( L^2 \) projection operator onto the finite element space. It is noted that \( (6) \) is a discrete analogue of the continuous maximal \( L^p \)-regularity \([14, 32]\) (also see \([20, \text{Lemma 2.1}]\))

\[ \| \partial_t \phi \|_{L^p(0,T;L^q)} + \| A\phi \|_{L^p(0,T;L^q)} \leq C_{p,q} \| f \|_{L^p(0,T;L^q)}, \quad 1 < p, q < \infty, \]

\[ \| \partial_t \phi \|_{L^p(0,T;W^{-1,q})} + \| \phi \|_{L^p(0,T;W^{1,q})} \leq C_{p,q} \| f \|_{L^p(0,T;W^{-1,q})}, \quad 1 < p, q < \infty. \]

In the last several decades, many efforts have been devoted to the maximal estimates of finite element solutions for linear parabolic equations. Most were based on the interior estimate/Green’s function approach developed in \([25, 26]\). Among these, the maximal \( L^p \)-regularity \([6]\) was proved in \([7]\) for the linear parabolic equation \([5]\) with time-independent coefficients in smooth domains. The corresponding \( L^\infty \) stability estimates have also been studied by many authors, see \([3, 15, 21, 22, 24, 25, 26, 27, 28, 29]\) and the references therein. All these works focused only on linear autonomous parabolic equations in a smooth domain with the diffusion coefficient being smooth enough, \( i.e. a_{ij} = a_{ij}(x) \in C^{2+\alpha}(\Omega) \). The discrete maximal \( L^p \)-regularity \([3, 7]\) in nonsmooth domains relies on more precise analysis of the discrete Green’s functions. The extension of maximal \( L^p \)-regularity to time discretization were made in \([16, 11, 12, 13]\). For linear equations with time-independent coefficients, the discrete maximal \( L^p \)-regularity was presented in our recent work \([20]\) for classical finite element solution in convex polyhedra and by Kemmochi and Saito \([12]\) for a lumped mass method in general polyhedra. The extension of \( L^p \) error estimate to a semilinear problem in a smooth domain was made in \([8]\), in which \( \sigma_{ij} = \sigma_{ij}(x) \) is assumed to be \( C^{2+\alpha}(\Omega) \) and time-independent. To extend the approach to general nonlinear problems, one of important issues to be considered is the regularity assumption of the diffusion coefficient. Recently, the regularity condition on the coefficients was weakened in \([17]\), and \([6, 8]\) were proved in \([17, 19]\) for the linear problem \([5]\) with time-independent and time-dependent Lipschitz continuous coefficients, respectively, with the Neumann boundary condition in smooth domains. A specific weakly nonlinear elliptic-parabolic system
from the model of incompressible miscible flow in porous media with the Neumann boundary condition in a smooth domain was also investigated [19]. The application of maximal $L^p$-regularity to time discretization of nonlinear parabolic equations can be found in [1, 2]. No analysis has been provided for finite element solutions of the nonlinear system with locally Lipschitz continuous coefficients in convex polyhedra.

In this paper, we focus our attention on the finite element solutions of (1) in convex polygons and polyhedra. To deal with the nonlinear problem, we first extend the discrete maximal $L^p$ estimates (6)-(8) for the autonomous case of the linear parabolic PDE (5), which was proved in [20], to the non-autonomous case with $a_{ij} = a_{ij}(x, t) \in L^\infty(0, T; W^{1, N+\epsilon_0}(\Omega)) \cap C(\Omega \times [0, T])$. Due to the nonsmoothness of the polyhedra, the extension is made with certain restrictions on $q$. By utilizing the proved $L^p$ estimates of the linear non-autonomous problem, we establish an optimal $L^p$ error estimate and an almost optimal $L^\infty$ error estimate of the finite element solution for the nonlinear parabolic equation. Our theoretical analysis provides a fundamental tool in establishing optimal error estimates of Galerkin FEMs for general nonlinear parabolic equations with coefficients of weak regularity.

We present our main results in the following two theorems.

**Theorem 1.1.** Let $\Omega$ be either a convex polygon in $\mathbb{R}^2$ or a convex polyhedron in $\mathbb{R}^3$, and assume that the coefficients $a_{ij}(x, t) \in L^\infty(0, T; W^{1, N+\epsilon_0}(\Omega)) \cap C(\Omega \times [0, T])$, $i, j = 1, \cdots, N$, satisfy the ellipticity condition

$$K_0^{-1}|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x, t)\xi_i\xi_j \leq K_0|\xi|^2, \quad \text{for } x \in \Omega \text{ and } t \in (0, T),$$

for some positive constant $\epsilon_0, K_0 > 0$. Then there exists $q_0 > 2$ such that the solutions of (4)-(5) satisfy

- (6) for $1 < p < \infty$ and $1 < q < q_0$,
- (7) for $1 < p < \infty$ and $1 < q < \infty$,
- (8) for $1 < p < \infty$ and $q_0' < q < \infty$,

where $q_0'$ is a positive integer satisfying $1/q_0' + 1/q_0 = 1$. The constant $q_0$ depends on $\epsilon_0$ and the interior angle of the corners/edges of the polygon/polyhedron.

**Remark 1.1** The restriction on the index $q$ is due to the smoothness of the domain and the time-dependency of the coefficients. Fortunately, both $p$ and $q$ can be arbitrarily close to infinity in the error estimate (8), which allows one to control the strong nonlinear terms involved in error analysis. If either the domain $\Omega$ is smooth or the coefficients $a_{ij}$ are independent of $t$, then (6) and (8) hold for all $1 < p, q < \infty$.

**Remark 1.2** For a two-dimensional convex polygon $\Omega$ with $a_{ij} \in L^\infty(0, T; W^{1, \infty}(\Omega)) \cap C(\overline{\Omega} \times [0, T])$, the constant $q_0$ can be chosen as $q_0 = 2/(2 - \min(\pi/\omega, 2))$, where
ω denotes the maximal interior angle of the convex polygon (if ω ∈ (0, π/2] then q₀ = ∞).

Remark 1.3  The results are presented for a nonlinear problem with very general assumptions on σᵢⱼ and g in (3). For example, the function g = ±eᵘ and even g = ±e|∇u|^² satisfy (3). With such strongly nonlinearities, the problem (1) may not have a globally smooth solution. Nevertheless, (1) always has a smooth solution for some short time interval [0, T] (local existence and uniqueness), provided that the initial data is smooth enough. In this article, we assume that the solution exists and sufficiently smooth (thus unique) in the time interval [0, T], and investigate the stability and convergence of the semi-discrete finite element solutions.

Theorem 1.2. Let Ω be either a convex polygon in ℝ² or a convex polyhedron in ℝ³. Assume that the condition (3) is satisfied and the solution of (1) satisfies

\[ u \in L^∞(0, T; W^{k+1,q}) \] for some fixed q > N and integer k ∈ [1, r]. Then there exists a positive constant h₀ such that when h < h₀ the solutions of (1)-(2) satisfy

\[ \|u - u_h\|_{L^p(0, T; L^q)} \leq C_p h^{k+1}, \quad \forall 2 < p < ∞, \] (13)
\[ \|u - u_h\|_{L^∞(0, T; L^q)} \leq C h^{k+1 - ε_h}, \] (14)

where εₜ → 0 as h → 0.

A traditional way to the optimal error estimate of Galerkin FEMs is based on an estimate in an energy-norm, i.e.

\[ \|u - u_h\|_{L^∞(0, T; L^2)} + \int_0^T \|u - u_h\|_{H^1}^2 dt \leq C h^{2r+2}. \] (15)

The main difficulty in such an approach for the general nonlinear equation (1) is the low regularity of σᵢⱼ(u, x) and the strong nonlinearities in g(u, ∇u, x). A well-known technique in the approach is to use the elliptic Ritz projection \( R_h \) : \( H_0^1(Ω) \) → \( S_h \) [6, 33] defined by (10). This approach requires the a priori estimate

\[ \|\partial_t(u - R_h u)\|_{L^2(Ω × (0, T))} \leq C h^{r+1}. \] (16)

The above estimate was established in [33] under the regularity assumption

\[ \|∇_x \partial_t σᵢⱼ(u(x, t), x)\|_{L^∞(Ω × (0, T))} \leq C \]

for a general nonlinear parabolic equation. The condition (16) was required when Nitsche’s trick (duality) was used in establishing (15). However, in some physical applications, the coefficients σᵢⱼ may not satisfy the regularity condition (16). One of examples is the incompressible miscible flow in porous media [5, 18, 30], where \( [σᵢⱼ]_{i,j=1}^N \) denotes the diffusion-dispersion tensor which is locally Lipschitz continuous in many cases. For σᵢⱼ being Lipschitz continuous, with a more precise analysis the above approach may yield a suboptimal error estimate

\[ \|u - u_h\|_{L^∞(0, T; L^2)} \leq C h^{r+1/2}, \] (17)
which is half-order lower than the optimal order. Instead of the elliptic Ritz projection, one may use the corresponding interpolation. However, the error estimate obtained is usually one order lower than the optimal one, except some special case

On the other hand, to deal with the strongly nonlinear term $g(u, \nabla u, x)$, the boundedness of $\|\nabla u_h\|_{L^\infty}$ is often needed in the error analysis since here, $g$ may not satisfy a Lipschitz condition. This boundedness is usually proved by using certain inverse inequality and the error estimate to be proved (in terms of mathematics induction or a truncation approach), i.e.

$$\|P_h u - u_h\|_{L^\infty(0,T;W^{1,\infty})} \leq Ch^{-1/2-N/2}$$

Since the above condition requires $r-1/2-N/2 > 0$ to control $\|P_h u - u_h\|_{L^\infty(0,T;W^{1,\infty})}$, the frequent-used linear and quadratic FEMs are excluded and $H^3$ regularity of the solution is required for higher-order methods.

The rest part of this paper is organized as follows. In Section 2, we introduce some notations to be used in this paper. We prove Theorem 1.1 and Theorem 1.2 in Section 3 and Section 4, respectively.

## 2. Notations and lemma

Let $W^{k,p}(\Omega)$ be the standard Sobolev space of functions defined in $\Omega$, where $k$ is any nonnegative integer and $1 \leq p \leq \infty$. Let $W_0^{1,p}(\Omega)$ be the subspace of $W^{1,p}(\Omega)$ consisting of functions whose traces vanish on $\partial \Omega$, and denote the dual space of $W_0^{1,p}(\Omega)$ by $W^{-1,p'}(\Omega)$ for $1 \leq p < \infty$. As conventions, we also use the notations $H^k(\Omega) := W^{k,2}(\Omega)$ and $L^p(\Omega) := W^{0,p}(\Omega)$.

For any Banach space $X$ and a given $T > 0$, $L^p(0,T;X)$ denotes the Bochner space equipped with the norm

$$\|f\|_{L^p(0,T;X)} = \begin{cases} \left( \int_0^T \|f(t)\|_X^p \, dt \right)^{1/p}, & 1 \leq p < \infty, \\ \text{ess sup}_{t \in (0,T)} \|f(t)\|_X, & p = \infty, \end{cases}$$

To simplify notations, we write $L^p$, $H^k$ and $W^{k,p}$ as the abbreviations of $L^p(\Omega)$, $H^k(\Omega)$ and $W^{k,p}(\Omega)$, respectively, and define

$$(\phi, \varphi) := \int_\Omega \phi(x) \varphi(x) \, dx.$$  

For a given $t \in (0,T)$, we use the notation $w(t)$ to denote the function $w(\cdot, t)$ defined on $\Omega$.

Let $a(x, t) = (a_{ij}(x, t))_{N \times N}$ be the coefficient matrix and define the operators

$$A(t) : H_0^1 \to H^{-1}, \quad A_h(t) : S_h \to S_h,$$

$$R_h(t) : H_0^1 \to S_h, \quad P_h : L^2 \to S_h,$$
by

\[(A(t)\phi, v) = (a(\cdot, t)\nabla\phi, \nabla v), \quad \text{for all } \phi, v \in H_0^1,\]

\[(A_h(t)\phi_h, v_h) = (a(\cdot, t)\nabla\phi_h, \nabla v_h), \quad \text{for all } \phi_h \in S_h \text{ and } v_h \in S_h,\]

\[(A_h(t)R_h(t)\phi, v_h) = (A(t)\phi, v_h), \quad \text{for all } \phi \in H_0^1 \text{ and } v_h \in S_h,\]

\[(P_h\phi, v_h) = (\phi, v_h), \quad \text{for all } \phi \in L^2 \text{ and } v_h \in S_h.\]

Clearly, $R_h(t)$ is the conventional Ritz projection operator associated to the elliptic operator $A(t)$ and $P_h$ is the $L^2$ projection operator onto the finite element space.

Under the assumptions of Theorem 1.1 there exists $q_0 > 2$ such that the following regularity estimates hold [9] [10]:

\[(w)_{L^q} \leq C_q\|\nabla \cdot (a\nabla w)\|_{L^r}, \quad \text{for any } 1 < q < q_0, \ \forall w \in H_0^1,\]

\[(\nabla w)_{L^\infty} \leq C_q\|\nabla \cdot (a\nabla w)\|_{L^r}, \quad \text{for any } q > N, \ \forall w \in H_0^1.\]

Moreover, we have the following projection error estimates:

\[(\phi - P_h\phi)_{W^{k,q}} \leq Ch^l\|\phi\|_{W^{l,q}}, \quad 1 \leq q \leq \infty, \ 0 \leq k \leq 1, \ k \leq l \leq r + 1,\]

\[(\phi - R_h\phi)_{W^{l,q}} \leq Ch^{l-1}\|\phi\|_{W^{l,q}}, \quad 1 < q \leq \infty, \ 1 \leq l \leq r + 1,\]

\[(\phi - R_h\phi)_{L^q} \leq Ch^l\|\phi\|_{W^{l,q}}, \quad q_0 < q < \infty, \ 1 \leq l \leq r + 1,\]

where (25)-(26) are standard error estimates of FEMs in convex polygons and polyhedra [3], and (27) follows a duality argument by using the regularity estimate (23).

Finally, we denote by $C = C_{p,q}$ a generic positive constant which is independent of $h$. To simplify the notations, we omit the subscripts in the generic constant when there is no ambiguity.

We present a generalized Gronwall's inequality in the following lemma, which was proved in [19].

**Lemma 2.1.** Let $1 < p < \infty$ and let $Y = Y(t)$ be a continuous function defined on $[0, T]$. If the function $Y(t)$ satisfies

\[\|Y\|_{L^p(\tau_1, \tau_2)} \leq \alpha\|Y\|_{L^1(\tau_1, \tau_2)} + \alpha Y(\tau_1) + \beta\]

for any $0 \leq \tau_1 < \tau_2 \leq s$ and $s \in (0, T]$, with some positive constants $\alpha$ and $\beta$, then we have

\[\|Y\|_{L^p(0, s)} \leq C_{T,\alpha,p}(Y(0) + \beta),\]

where the constant $C_{T,\alpha,p}$ is independent of $s \in (0, T]$.

3. **Proof of Theorem 1.1**

In this section, we study the linear parabolic equation (5) and the corresponding FE solution and prove Theorem 1.1. The following lemma was presented in [20] and will be used in our proof.
Lemma 3.1. Under the assumptions of Theorem 1.1, if the coefficients $a_{ij} = a_{ij}(x)$, $i, j = 1, \cdots, N$, are independent of the time $t$, then Theorem 1.1 holds.

To prove Theorem 1.1, firstly we assume that $\phi^0_h = \phi^0 = 0$ and prove (28). For this purpose, we partition the time interval $[0, T]$ uniformly into $0 = t_0 < t_1 < \cdots < t_{N_0} = T$, with $t_n - t_{n-1} = \Delta t = T/N_0$, for $1 \leq n \leq N_0$, and rewrite (4) as

$$
\begin{align*}
\partial_t \phi_h(t) + A_h(t)\phi_h(t) &= P_h f(t), \\
\phi_h(0) &= 0.
\end{align*}
$$

From the equation above we can see that the function $\psi_h(t) = P_h (\phi(t) - \phi_h(t))$ satisfies

$$
\begin{align*}
\partial_t \psi_h(t) + A_h(t)\psi_h(t) &= (A_h(t_n) - A_h(t))\psi_h(t) + A_h(t)(P_h \phi(t) - R_h(t)\phi(t)), \\
\psi_h(0) &= 0.
\end{align*}
$$

We extend $\psi_h$ to be zero for $t \leq 0$. If we let $\varphi^0_h(t) = \psi_h(t) - \psi_h(2t_n - t)$ for $t \in [t_n, t_{n+1}]$, then $\varphi^0_h$ is the solution of the equation

$$
\begin{align*}
\partial_t \varphi^0_h(t) + A_h(t)\varphi^0_h(t) &= (A_h(t_n) - A_h(t))\psi_h(t) + A_h(t)(P_h \phi(t) - R_h(t)\phi(t)), \\
&= (A_h(t_n) - A_h(t))\psi_h(t) + A_h(t)(P_h \phi(t) - R_h(t)\phi(t)) - A_h(t)\psi_h(2t_n - t) - \partial_t \psi_h(2t_n - t), \\
&= A_h(2t_n - t)\psi_h(t) + A_h(2t_n - t)(P_h \phi(t) - R_h(t)\phi(t)) - A_h(2t_n - t)\psi_h(t) - \partial_t \psi_h(t)
\end{align*}
$$

in the time interval $[t_n, t_{n+1}]$, with $\varphi^0_h(t_n) = 0$. This is a parabolic equation with time-independent coefficients. In view of Lemma 3.1, we can apply (7) to the equation above in the time interval $[t_n, t_{n+1}]$ to get

$$
\|\varphi^0_h\|_{L^p(t_n, t_{n+1}; W^{1,p})} \leq C^* \left( \sup_{t \in [t_n, t_{n+1}]} \|a_{ij}(t_n) - a_{ij}(t)\|_{L^\infty} \|\psi_h\|_{L^p(t_n, t_{n+1}; W^{1,p})} \right. 
$$

$$
\left. + \|P_h \phi - R_h \phi\|_{L^p(t_n, t_{n+1}; W^{1,p})} + \|\psi_h\|_{L^p(t_n, t_{n+1}; W^{1,p})} \right)
$$

where the constant $C^*$ is independent of $\Delta t$. Since $a_{ij} \in C(\Omega \times [0, T])$, there exists a positive constant $\tau_0$ such that $C^* \sup_{t \in [t_n, t_{n+1}]} \|a_{ij}(t_n) - a_{ij}(t)\|_{L^\infty} < 1/2$ when $\Delta t < \tau_0$ and therefore, the last inequality reduces to

$$
\|\psi_h\|_{L^p(t_n, t_{n+1}; W^{1,p})} \leq C\|\psi_h\|_{L^p(t_n, t_{n+1}; W^{1,p})} + C\|P_h \phi - R_h \phi\|_{L^p(t_n, t_{n+1}; W^{1,p})} + C\|\psi_h\|_{L^p(t_n, t_{n+1}; W^{1,p})} 
$$

for $n = 1, 2, \ldots, N_0 - 1$. Note that the number of iterations is bounded by the constant $[T/\tau_0] + 1$, where $[T/\tau_0]$ denotes the largest integer which does not exceed $T/\tau_0$. Iterating the above inequality leads to

$$
\|\psi_h\|_{L^p(0, T; W^{1,p})} \leq C\|P_h \phi - R_h \phi\|_{L^p(0, T; W^{1,p})},
$$
which further implies (by the definition of $\psi_h$)

$$\|\phi_h\|_{L^p(0,T;W^{1,1})} \leq C\|\psi_h\|_{L^p(0,T;W^{1,1})} + C\|P_h\phi\|_{L^p(0,T;W^{1,1})}$$

$$\leq C\|P_h\phi - R_h\phi\|_{L^p(0,T;W^{1,1})} + C\|P_h\phi\|_{L^p(0,T;W^{1,1})}$$

$$\leq C\|\phi\|_{L^p(0,T;W^{1,1})}$$

(30)

where we have used (12) and (26) with (30) the equation (28), we obtain

$$\|A_h\|_{L^p(0,T;W^{1,1})} \leq C\|f\|_{L^p(0,T;W^{1,1})},$$

(31)

As a consequence of (29)-(32), we see that

$$\|A_h\|_{L^p(0,T;L^q)} \leq \|A_h\|_{L^p(0,T;L^q)} + C\|\psi_h\|_{L^p(0,T;L^q)}$$

(32)

As a consequence of (29)-(32), we see that

$$\|A_h\phi_h\|_{L^p(0,T;L^q)} \leq \|A_h\phi_h\|_{L^p(0,T;L^q)} + C\|\phi_h\|_{L^p(0,T;L^q)}$$

(33)

which gives (via duality)

$$\|\partial_t\phi_h\|_{W^{-1,1}} \leq C(\|f\|_{W^{-1,1}} + \|\phi_h\|_{W^{1,1}}),$$

and so

$$\|\partial_t\phi_h\|_{L^p(0,T;W^{-1,1})} \leq C(\|f\|_{L^p(0,T;W^{-1,1})} + \|\phi_h\|_{L^p(0,T;W^{1,1})})$$

(34)

This completes the proof of (7).

To prove (6), we note that

$$\|(A_h(t)\psi_h(t), v)\| = \|(A_h(t)\psi_h(t), P_hv)\|$$

$$= \|(a(\cdot,t)\nabla\psi_h(t), \nabla P_hv)\|$$

$$\leq C\|\psi_h(t)\|_{W^{1,1}}\|P_hv\|_{W^{1,1}}$$

$$\leq Ch^{-1}\|\psi_h(t)\|_{W^{1,1}}\|P_hv\|_{L^q}$$

(35)

which gives (via duality)

$$\|A_h(t)\psi_h(t)\|_{L^q} \leq Ch^{-1}\|\psi_h(t)\|_{W^{1,1}}.$$

(36)
By using (23), which requires $1 < q < q_0$, the last inequality further reduces to
\begin{equation}
\|A_h \phi_h\|_{L^p(0,T;L^q)} \leq \|A_h P_h \phi\|_{L^p(0,T;L^q)} + C \|f\|_{L^p(0,T;L^q)}.
\end{equation}

To estimate $\|A_h P_h \phi\|_{L^p(0,T;L^q)}$, we consider
\begin{equation}
|\langle A_h(t) P_h \phi(t), v \rangle| = |\langle (A_h(t) P_h \phi(t), P_h v) \rangle| 
\leq |\langle (A_h(t)(P_h \phi(t) - R_h(t)) \phi(t), P_h v) \rangle| + |\langle A(t) \phi(t), P_h v \rangle| \quad \text{(by using (21))}
\leq C \|P_h \phi(t) - R_h(t) \phi(t)\|_{W^{1,q}} \|P_h v\|_{W^{1,q'}} + \|A(t) \phi(t)\|_{L^q} \|P_h v\|_{L^{q'}}
\leq C \|\phi(t)\|_{W^{1,q}} \|P_h v\|_{W^{1,q'}} + C \|A(t) \phi(t)\|_{L^q} \|P_h v\|_{L^{q'}}
\leq C \|\phi(t)\|_{W^{1,q}} \|P_h v\|_{W^{1,q'}} + \|A(t) \phi(t)\|_{L^q} \|P_h v\|_{L^{q'}}. \quad \text{(by using (25))}
\end{equation}

By duality, the last inequality implies
\begin{equation}
\|A_h(t) P_h \phi(t)\|_{L^p(0,T;L^q)} \leq C \|\phi(t)\|_{L^p(0,T;W^{2,q})} + \|A(t) \phi(t)\|_{L^p(0,T;L^q)} \leq C \|f\|_{L^p(0,T;L^q)},
\end{equation}
where we have used (11) and (23), which requires $1 < q < q_0$. From the above inequality and (33), we get
\begin{equation}
\|A_h \phi_h\|_{L^p(0,T;L^q)} \leq C \|f\|_{L^p(0,T;L^q)}, \quad \forall 1 < p < \infty, \ 1 < q < q_0.
\end{equation}

By using the above inequality, we see from (28) that
\begin{equation}
\|\partial_t \phi_h\|_{L^p(0,T;L^q)} \leq \|A_h \phi_h\|_{L^p(0,T;L^q)} + \|P_h f\|_{L^p(0,T;L^q)} \leq C \|f\|_{L^p(0,T;L^q)}
\end{equation}
for $1 < p < \infty$ and $1 < q < q_0$. We complete the proof of (6).

Secondly, we drop the assumption $\phi_h^0 = \phi^0 = 0$ and prove (8). For this purpose, we consider $e_h(t) = P_h \phi(t) - \phi_h(t) + \phi_h^0 - P_h \phi_0$, which is the solution of the equation
\begin{equation}
\begin{cases}
\partial_t e_h(t) + A_h(t) e_h(t) = A_h(t) g_h(t), \\
\phi_h(0) = 0.
\end{cases}
\end{equation}

with
\begin{equation}
ge_h(t) = \phi_h^0 - P_h \phi_0 + P_h \phi(t) - R_h(t) \phi(t).
\end{equation}

By applying (7) to the above equation, we derive
\begin{equation}
\|\partial_t e_h\|_{L^p(0,T;W^{1,q})} + \|e_h\|_{L^p(0,T;W^{1,q})} \leq C \|g_h\|_{L^p(0,T;W^{1,q})}.
\end{equation}

Moreover, let $v$ be the solution of the backward parabolic equation
\begin{equation}
\begin{cases}
\partial_t v + \nabla \cdot (a \nabla v) = -\varphi \quad \text{in } \Omega, \\
v = 0 \quad \text{on } \partial \Omega, \\
v(T) = 0,
\end{cases}
\end{equation}
which obeys the basic estimate (the same as the forward parabolic equation)
\begin{equation}
\|v\|_{L^{q'}(0,T;W^{2,q'})} \leq C \|\varphi\|_{L^{q'}(0,T;L^{q'})}.
\end{equation}
The last inequality is a consequence of (11) and (23), which requires \( q_0 < q < \infty \).

From (39), we see that
\[
\int_0^T (e_h, \varphi) \, dt
\]
\[
= \int_0^T (e_h, -\partial_t v - \nabla \cdot (a \nabla v)) \, dt
\]
\[
= \int_0^T \left[ (\partial_t e_h, v) + (a \nabla e_h, \nabla v) \right] \, dt
\]
\[
= \int_0^T \left[ (\partial_t e_h(t), v(t) - P_h v(t)) + (A_h(t)e_h, v(t) - P_h v(t)) \right] \, dt + \int_0^T (A_h(t)g_h(t), P_h v(t)) \, dt
\]
\[
= \int_0^T (A_h(t)e_h(t), R_h(t)v(t) - P_h v(t)) \, dt + \int_0^T (g_h(t), A(t)v(t)) \, dt
\]
\[
+ \int_0^T (g_h(t), A_h(t)(P_h v(t) - R_h(t)v(t))) \, dt
\]
\[
\leq C \| e_h \|_{L^p(0,T;W^{1,q})} \| R_h v - P_h v \|_{L^{q'}(0,T;W^{1,q'})} + \| g_h \|_{L^p(0,T;L^q)} \| A v \|_{L^{q'}(0,T;L^{q'})}
\]
\[
+ C \| g_h \|_{L^p(0,T;W^{1,q})} \| R_h v - P_h v \|_{L^{q'}(0,T;W^{1,q'})}
\]
\[
\leq C h \| e_h \|_{L^p(0,T;W^{1,q})} \| v \|_{L^{q'}(0,T;W^{1,q'})} + \| g_h \|_{L^p(0,T;L^q)} \| v \|_{L^{q'}(0,T;W^{1,q'})}
\]
\[
\leq C (h \| e_h \|_{L^p(0,T;W^{1,q})} + \| g_h \|_{L^p(0,T;L^q)}) \| \varphi \|_{L^{q'}(0,T;L^{q'})}.
\]

By duality and using (38), we derive that
\[
\| e_h \|_{L^p(0,T;L^q)}
\]
\[
\leq C(h \| e_h \|_{L^p(0,T;W^{1,q})} + \| g_h \|_{L^p(0,T;L^q)})
\]
\[
\leq C(h \| g_h \|_{L^p(0,T;W^{1,q})} + \| P_h \phi^0 - \phi_h \|_{L^{q'}} + \| g_h \|_{L^p(0,T;L^q)})
\]
\[
\leq C(\| P_h \phi - R_h \phi \|_{L^p(0,T;L^q)} + \| P_h \phi^0 - \phi_h \|_{L^{q'}})
\]
\[
\leq C(\| P_h \phi \|_{L^p(0,T;L^q)} + \| P_h \phi^0 \|_{L^{q'}})
\]

This proves (38).

The proof of Theorem 1.1 is completed.

4. Proof of Theorem 1.2

In this section, we study the nonlinear parabolic equation (1) and the corresponding FE solution and prove Theorem 1.2 by applying Theorem 1.1.

Let \( M = \| u \|_{L^\infty(\Omega_T)} + \| \nabla u \|_{L^\infty(\Omega_T)} + \| P_h u \|_{L^\infty(\Omega_T)} + \| \nabla (P_h u) \|_{L^\infty(\Omega_T)} + 2 \), and let \( I_c = \{ x \in \mathbb{R} : |x| \leq c \} \) for any \( c > 0 \). Let \( \zeta(u, w) \) be a smooth function...
defined on $\mathbb{R} \times \mathbb{R}^N$ such that $\zeta(u, w) = 1$ for $(u, w) \in I_M \times I_M^N$ and $\zeta(u, w) = 0$ for $(u, w) \notin I_M \times I_M^N$. We define the truncated coefficients and right-hand side by
\[
\sigma_{ij}(u, x) = \begin{cases} 
\sigma_{ij}(u, x) & \text{if } -2M \leq u \leq 2M, \\
\sigma_{ij}(2M, x) & \text{if } u \geq 2M, \\
\sigma_{ij}(-2M, x) & \text{if } u \leq -2M,
\end{cases}
\]
\[
g(u, w, x) = g(u, w, x)\zeta(u, w),
\]
and define a cut-off function by
\[
\chi(s) = \begin{cases} s, & \text{if } |s| \leq 2, \\
2, & \text{if } s \geq 2, \\
-2, & \text{if } s \leq -2.
\end{cases}
\]

Based on the assumption (3), the truncated functions satisfy
\[
|\sigma_{ij}(u, x)| + |\partial_u \sigma_{ij}(u, x)| + |\partial_x \sigma_{ij}(u, x)| \leq K_{2M},
\]
\[
K_{2M}^{-1}|\xi|^2 \leq \sum_{i,j=1}^N \sigma_{ij}(u, x, t)\xi_i\xi_j \leq K_{2M}|\xi|^2,
\]
\[
|\mathcal{G}(u, \eta, x)| + |\partial_u \mathcal{G}(u, \eta, x)| + |\partial_\eta \mathcal{G}(u, \eta, x)|
+ |\partial_{u, \eta}^2 \mathcal{G}(u, \eta, x)| + |\partial_{u, \eta, \eta}^2 \mathcal{G}(u, \eta, x)| \leq CK_{2M}
\]
for any $u \in \mathbb{R}$ and $\eta \in \mathbb{R}^N$. Let $\theta_h = u - P_h u$ be the projection error and let $e_h = u_h - P_h u$. By comparing (1) and (2), it is easy to see that $e_h$ is the solution of the following finite element equation
\[
\begin{align*}
(\partial_t e_h, v) &+ \sum_{i,j=1}^N (\sigma_{ij}(u, x)\partial_i e_h, \partial_j v) \\
&= \sum_{i,j=1}^N (\sigma_{ij}(u, x)\partial_i \theta_h, \partial_j v) \\
&\quad + \sum_{i,j=1}^N ((\sigma_{ij}(u, x) - \sigma_{ij}(e_h + P_h u, x))\partial_i (e_h + P_h u), \partial_j v) \\
&\quad - (g(P_h u, \nabla P_h u, x) - g(e_h + P_h u, \nabla (e_h + P_h u), x), v) \\
&\quad + (g(P_h u, \nabla P_h u, x) - g(u, \nabla u, x), v),
\end{align*}
\]
(42) for all $v \in S_h$. 
Also we let \( \varphi_h \in S_h \) be the solution of the following nonlinear finite element equation:

\[
(\partial_t \varphi_h, v) + \sum_{i,j=1}^{N} (\sigma_{ij}(u, x) \partial_i \varphi_h, \partial_j v) = \sum_{i,j=1}^{N} (\sigma_{ij}(u, x) \partial_i \varphi_h, \partial_j v) \\
+ \sum_{i,j=1}^{N} (\sigma_{ij}(u, x) - \sigma_{ij}(\varphi_h + P_h u, x))(\partial_i u + \chi(\varphi_h + P_h u - u)) \partial_j v)
- (\mathcal{G}(P_h u, \nabla P_h u, x) - \mathcal{G}(\varphi_h + P_h u, \nabla(\varphi_h + P_h u), x), v)
\]

(43)

with the initial condition \( \varphi_h(0) = I_h u(0) - P_h u(0) \). With the spatial discretization, (43) is essentially a system of ordinary differential equations. Thus existence and uniqueness of solutions of (43) are obvious and the solution is continuous with respect to \( t \in [0, T] \). Moreover, if the inequality holds for \( t \) then

\[
\|\nabla(\varphi_h + P_h u)\|_{L^\infty} \leq M, \|\nabla(\varphi_h + P_h u)\|_{L^\infty} \leq M, \text{ which imply}
\]

\[
\mathcal{G}(\varphi_h + P_h u, \nabla(\varphi_h + P_h u), x) = g(\varphi_h + P_h u, \nabla(\varphi_h + P_h u), x).
\]

In this case, (43) reduces to (42) and so \( \varphi_h = e_h \).

Now we proceed with a mathematical induction on

(44) \[ |\varphi_h| < 1/2, \quad |\nabla \varphi_h| < 1/2. \]

Since

\[
|\varphi_h(0)| + |\nabla \varphi_h(0)| = |I_h u(0) - P_h u(0)| + |\nabla(I_h u(0) - P_h u(0))| \leq C h^{1-N/q} \|u\|_{W^{2,q}}
\]

for \( q > N \), there exists a positive constant \( h_1 \) such that when \( h < h_1 \) the inequality (44) holds at the initial time \( t = 0 \) and

(45) \[ \|P_h u - u\|_{L^\infty(0,T;L^\infty)} < 1/2, \quad \|\nabla(P_h u - u)\|_{L^\infty(0,T;L^\infty)} < 1/2. \]

By continuity, we can assume that (44) holds for \( t \in [0, s_0] \) for some \( 0 < s_0 \leq T \). Moreover, if the inequality holds for \( t \in [0, s] \) with some \( 0 < s \leq T \), then there exists a positive constant \( \delta_h \) such that

(46) \[ |\varphi_h| \leq 1, \quad |\nabla \varphi_h| \leq 1, \quad t \in [0, s + \delta_h] \cap [0, T], \]

(47) \[ \|\varphi_h + P_h u - u\|_{L^\infty(0,s+\delta_h;L^\infty)}, \quad \|\partial_t(\varphi_h + P_h u - u)\|_{L^\infty(0,s+\delta_h;L^\infty)} \leq 3/2, \]

which implies

\[ \varphi_h = e_h = u_h - P_h u, \quad \text{for } t \in [0, s + \delta_h]. \]
In the following, we shall prove that (44) also holds for $t \in [0, s + \delta_h]$. Then by induction, (44) holds in $[0, T]$ or an open subset of $[0, T]$, and by continuity, the latter can be extended to the closure of this subset. This implies that (44) holds for all $t \in [0, T]$.

To prove (44) for $t \in [0, s + \delta_h]$, we need to estimate those terms in the right-hand side of (43). To make use of Lemma 2.1, we consider (43) in the time interval $(\tau_1, \tau_2)$, with $0 \leq \tau_1 < \tau_2 \leq s + \delta_h$. It should be kept in mind that the generic constant $C$ below should be independent of $s$ and $\delta_h$ (but may depend on $T$). By noting the fact

\[ (\mathcal{G}(\bar{e}_h + P_h u, \nabla(\bar{e}_h + P_h u), x), v) = (\mathcal{G}(u_h, \nabla u_h, x), v), \]

we have

\[
(\mathcal{G}(P_h u, \nabla P_h u, x) - \mathcal{G}(u, \nabla u, x), v) = (b_1 e_h + b_2 \cdot \nabla e_h, v),
\]

\[
(\mathcal{G}(P_h u, \nabla P_h u, x) - \mathcal{G}(u, \nabla u, x))
= (\mathcal{G}(P_h u, \nabla P_h u, x) - \mathcal{G}(u, \nabla P_h u, x) + \mathcal{G}(u, \nabla P_h u, x) - \mathcal{G}(u, \nabla u, x), v)
= (b_3 \theta_h, v) - \left( \nabla \theta_h \cdot \int_0^1 \nabla w \mathcal{G}(u, (1 - s)\nabla P_h u + s \nabla u, x) ds, v \right)
= (b_3 \theta_h, v) - \left( \nabla \theta_h \cdot \int_0^1 [\nabla w \mathcal{G}(u, (1 - s)\nabla P_h u + s \nabla u, x) - \nabla w \mathcal{G}(u, \nabla u, x)] ds, v \right)
- \left( \nabla \cdot (\theta_h \nabla w \mathcal{G}(u, \nabla u, x)) - \theta_h \nabla \cdot (\nabla w \mathcal{G}(u, \nabla u, x)), v \right)
= (b_3 \theta_h, v) + \left( B \nabla \theta_h \cdot \nabla \theta_h + \theta_h \nabla \cdot b_4, v \right) + (b_4 \theta_h, \nabla v),
\]

and

\[
\sum_{i,j=1}^N \left( (\sigma_{ij}(u, x) - \sigma_{ij}(u_h, x))(\partial_i u + \chi(\partial_i (u_h - u))), \partial_j v = \left( (\theta_h - e_h), b_5 \cdot \nabla v, \right),
\]

\[
\sum_{i,j=1}^N \left( (\sigma_{ij}(u, x) - \sigma_{ij}(u_h, x))(\partial_i u + \chi(\partial_i (u_h - u))), \partial_j v = \left( (\theta_h - e_h), b_5 \cdot \nabla v, \right),
\]
where
\[
b_1 = - \int_0^1 \partial_u \overline{g}((1 - s)P_h u + s u_h + (1 - s)\nabla P_h u + s \nabla u_h, x) \, ds,
\]
\[
b_2 = \int_0^1 \partial_w \overline{g}((1 - s)P_h u + s u_h, (1 - s)\nabla P_h u + s \nabla u_h, x) \, ds,
\]
\[
b_3 = - \int_0^1 \partial_u \overline{g}((1 - s)P_h u + s u, \nabla P_h u, x) \, ds,
\]
\[
b_4 = \nabla_w \overline{g}(u, \nabla u, x),
\]
\[
B = - \int_0^1 \int_0^1 \nabla_u \overline{g}(u, (1 - s')(1 - s)\nabla u + s\nabla P_h u + s' \nabla u, x) \, ds \, ds',
\]
\[
b_5 \cdot \nabla v = \sum_{i,j=1}^N \partial_j v(\partial_i u + \chi(\partial_i(u_h - u))) \int_0^1 \partial_u \overline{\sigma}_{ij}((1 - s)u_h + su, x) \, ds.
\]

With above formulas, (43) reduces to
\[
(\partial_t (e_h - \theta_h), v) + \sum_{i,j=1}^N (\overline{\sigma}_{ij}(u, x) \partial_i (e_h - \theta_h) \partial_j v)
\]
\[
= (\theta_h, (b_4 + b_5) \cdot \nabla v) + (b_3 \theta_h + B \nabla \theta_h \cdot \nabla \theta_h + \theta_h \nabla \nabla \cdot b_4, v)
\]
\[
- (b_2 \cdot \nabla e_h + b_1 e_h, v) - (e_h, b_5 \cdot \nabla v), \quad \forall v \in S_h,
\]
(48)
where we have noted \((\partial_t \theta_h, v) = 0\). Moreover, we can see
\[
(\partial_t (e_h + P_h u - w_h - u), v)
\]
\[
+ \sum_{i,j=1}^N (\overline{\sigma}_{ij}(u, x) \partial_i (e_h + P_h u - w_h - u) \partial_j v) = 0, \quad \forall v \in S_h,
\]
(49)
where \(w_h\) is the finite element solution of the equation
\[
(\partial_t w_h, v) + \sum_{i,j=1}^N (\overline{\sigma}_{ij}(u, x) \partial_i w_h, \partial_j v)
\]
\[
= (\theta_h, (b_4 + b_5) \cdot \nabla v) + (b_3 \theta_h + B \nabla \theta_h \cdot \nabla \theta_h + \theta_h \nabla \nabla \cdot b_4, v)
\]
\[
- (b_2 \cdot \nabla e_h + b_1 e_h, v) - (e_h, b_5 \cdot \nabla v)
\]
\[
= (\theta_h, (b_4 + b_5) \cdot \nabla v) + (b_3 \theta_h + B \nabla \theta_h \cdot \nabla \theta_h + \theta_h \nabla \nabla \cdot b_4, v)
\]
\[
- ((b_1 - \nabla \cdot b_2) e_h, v) - (e_h, (b_5 - b_2) \cdot \nabla v), \quad \forall v \in S_h,
\]
(50)
with the initial condition \(w_h(\tau_1) = 0\).
By (46), it is easy to see that
\[ \| b_3 \|_{L^\infty(0,s+\delta_h;L^\infty)} + \| b_4 \|_{L^\infty(0,s+\delta_h;L^\infty)} + \| b_5 \|_{L^\infty(0,s+\delta_h;L^\infty)} \]
\[ + \| \nabla \cdot b_2 \|_{L^\infty(0,s+\delta_h;L^\infty)} + \| \nabla \cdot b_4 \|_{L^\infty(0,s+\delta_h;L^\infty)} + \| B \|_{L^\infty(0,s+\delta_h;L^\infty)} \leq C \]
for \( t \in [0,s+\delta_h] \) and also we have
\[ \| \theta_h \|_{L^p(0,T;L^q)} \leq Ch^m \| u \|_{L^p(0,T;W^{m,q})}, \quad 0 \leq m \leq r + 1, \quad 1 \leq p, q \leq \infty. \]

Substituting \( v = w_h \) into (50) gives
\[ \| w_h \|_{L^\infty(\tau_1,\tau_2;L^2)} \leq C \| \theta_h \|_{L^2(\tau_1,\tau_2;L^2)} + C \| \nabla \theta_h \|_{L^2(\tau_1,\tau_2;L^2)} + C \| w_h \|_{L^2(\tau_1,\tau_2;L^2)} \]
\[ \leq C \| \theta_h \|_{L^2(\tau_1,\tau_2;L^2)} + C \| \nabla \theta_h \|_{L^2(\tau_1,\tau_2;L^2)} + C \| w_h \|_{L^2(\tau_1,\tau_2;L^2)} \]
\[ \leq Ch^{k+1} \| u \|_{L^2(\tau_1,\tau_2;H^{k+1})} + Ch^{2k} \| u \|_{L^2(\tau_1,\tau_2;W^{k+1,q})} + C \| \theta_h \|_{L^2(\tau_1,\tau_2;L^2)} \]
\[ \leq Ch^{k+1} + C \| \theta_h \|_{L^2(\tau_1,\tau_2;L^2)}. \]

By applying Theorem 1.1 to (50), we derive
\[ \| w_h \|_{L^p(\tau_1,\tau_2;W^{1,q})} \leq C \| \theta_h \|_{L^p(\tau_1,\tau_2;L^q)} + C \| \nabla \theta_h \|_{L^p(\tau_1,\tau_2;L^q)} + C \| w_h \|_{L^p(\tau_1,\tau_2;L^q)} \]
\[ \leq C \| \theta_h \|_{L^p(\tau_1,\tau_2;L^q)} + C \| \nabla \theta_h \|_{L^p(\tau_1,\tau_2;L^q)} + C \| w_h \|_{L^p(\tau_1,\tau_2;L^q)} \]
\[ \leq Ch^{k+1} \| u \|_{L^p(\tau_1,\tau_2;W^{k+1,q})} + Ch^{2k} \| u \|_{L^p(\tau_1,\tau_2;W^{k+1,q})} + C \| \theta_h \|_{L^p(\tau_1,\tau_2;L^q)} \]
\[ \leq Ch^{k+1} + C \| \theta_h \|_{L^p(\tau_1,\tau_2;L^q)}, \]
and by the Sobolev interpolation inequality we have (for \( q > N \))
\[ \| w_h \|_{L^p(\tau_1,\tau_2;L^q)} \leq C \| w_h \|_{L^p(\tau_1,\tau_2;L^2)}^{2/q} \| w_h \|_{L^p(\tau_1,\tau_2;L^q)}^{1-2/q} \]
\[ \leq C \| w_h \|_{L^p(\tau_1,\tau_2;L^2)}^{2/q} \| w_h \|_{L^p(\tau_1,\tau_2;L^2)}^{1-2/q} \]
\[ \leq C (Ch^{k+1} + C \| \theta_h \|_{L^p(\tau_1,\tau_2;L^2)}^{2/q} (Ch^{k+1} + C \| \theta_h \|_{L^p(\tau_1,\tau_2;L^2)})^{1-2/q} \]
\[ \leq Ch^{k+1} + C \| \theta_h \|_{L^2(\tau_1,\tau_2;L^2)} + C \| \theta_h \|_{L^p(\tau_1,\tau_2;L^2)}. \]

Again, applying Theorem 1.1 to (49), we obtain
\[ \| e_h + P_h u - w_h - u \|_{L^p(\tau_1,\tau_2;L^q)} \]
\[ \leq C \| e_h(\tau_1) + P_h u(\tau_1) - u(\tau_1) \|_{L^q} + C \| u - P_h u \|_{L^p(\tau_1,\tau_2;L^q)} + C \| u - R_h u \|_{L^p(\tau_1,\tau_2;L^q)} \]
\[ \leq C \| e_h(\tau_1) \|_{L^q} + Ch^{k+1} \| u \|_{L^\infty(\tau_1,\tau_2;W^{k+1,q})}, \]
(51)
and therefore,

\[
\| e_h \|_{L^p(\tau_1, \tau_2; \mathbb{L}^q)} \leq \| e_h + P_h u - w_h - u \|_{L^p(\tau_1, \tau_2; \mathbb{L}^q)} \\
+ \| u - P_h u \|_{L^p(\tau_1, \tau_2; \mathbb{L}^q)} + \| w_h \|_{L^p(\tau_1, \tau_2; \mathbb{L}^q)} \\
\leq C\| e_h(\tau_1) \|_{\mathbb{L}^q} + C\| h \|^{k+1} + \| w_h \|_{L^p(\tau_1, \tau_2; \mathbb{L}^q)} \\
\leq C\| e_h(\tau_1) \|_{\mathbb{L}^q} + C\| h \|^{k+1} + C\| e_h \|_{L^2(\tau_1, \tau_2; \mathbb{L}^q)} + C\| \| e_h \|_{L^{2n}\mathbb{L}^q(\tau_1, \tau_2; \mathbb{L}^q)} \\
\leq C\| e_h(\tau_1) \|_{\mathbb{L}^q} + C\| h \|^{k+1} + \| e_h \|_{L^p(\tau_1, \tau_2; \mathbb{L}^q)} + C\| e_h \|_{L^p(\tau_1, \tau_2; \mathbb{L}^q)}
\]

for any \( \epsilon \in (0, 1) \), which further leads to

\[
\| e_h \|_{L^p(\tau_1, \tau_2; \mathbb{L}^q)} \leq C\| e_h(\tau_1) \|_{\mathbb{L}^q} + C\| h \|^{k+1} + C\| e_h \|_{L^1(\tau_1, \tau_2; \mathbb{L}^q)},
\]

By Lemma 2.1 together with the above inequality, we arrive at the following estimate

\[
\| e_h \|_{L^p(0, s+\delta_h; \mathbb{L}^q)} \leq C\| e_h(0) \|_{\mathbb{L}^q} + C\| h \|^{k+1} \leq C\| h \|^{k+1},
\]

where the constant \( C \) is independent of \( s \) and \( \delta_h \).

Note that \( e_h \) is the solution of the equation

\[
\partial_t e_h + \overline{\mathcal{A}} h e_h = f_h,
\]

where the operator \( \overline{\mathcal{A}} h \) and the function \( f_h \) are defined by

\[
(\overline{\mathcal{A}} h(t) \phi_h, v) = (\nabla(u, x, t) \nabla \phi_h, \nabla v) \quad \text{for all } \phi_h \in S_h, \ v \in S_h,
\]

\[
f_h = \overline{\mathcal{A}} h \theta_h - \nabla_h \cdot [(b_4 + b_5) \theta_h + (b_2 - b_3) e_h] \\
+ \eta_h [b_2 \theta_h + B \nabla \theta_h \cdot \nabla h + \theta_h \nabla \cdot b_4 + ((\nabla \cdot b_2) - b_1) \theta h].
\]

Since

\[
\| \overline{\mathcal{A}} h \theta_h \|_{L^p(0, s+\delta_h; W^{-1, q})} \leq C\| \theta_h \|_{L^p(0, s+\delta_h; W^{-1, q})} \leq C\| h \|^{k},
\]

\[
\| \nabla_h \cdot [(b_4 + b_5) \theta_h + (b_2 - b_3) e_h] \|_{L^p(0, s+\delta_h; W^{-1, q})} \leq C\| (b_4 + b_5) \theta_h + (b_2 - b_3) e_h \|_{L^p(0, s+\delta_h; W^{-1, q})} \leq C\| h \|^{k+1},
\]

\[
\| B \nabla \theta_h \cdot \nabla \theta_h \|_{L^p(0, s+\delta_h; W^{-1, q})} \leq C\| B \nabla \theta_h \cdot \nabla \theta_h \|_{L^p(0, s+\delta_h; L^q)} \leq C\| \nabla \theta_h \|_{L^p(0, s+\delta_h; L^q)} \leq C\| h \|^{k+1},
\]

it follows that \( \| f_h \|_{L^p(0, s+\delta_h; W^{-1, q})} \leq C\| h \|^{k} \). By Theorem 1.1 we have

\[
\| \partial_t e_h \|_{L^p(0, s+\delta_h; W^{-1, q})} + \| e_h \|_{L^p(0, s+\delta_h; W^{1, q})} \leq C\| f_h \|_{L^p(0, s+\delta_h; W^{-1, q})} \leq C\| h \|^{k}.
\]

Since \( k \geq 1 \), with an inverse inequality we have

\[
\| \partial_t e_h \|_{L^p(0, s+\delta_h; L^q)} \leq C\| h \|^{k-1} \| \partial_t e_h \|_{L^p(0, s+\delta_h; W^{-1, q})} \leq C.
\]
and by the Sobolev interpolation inequality,
\[ \|e_h\|_{L^\infty(0,s+\delta_h;L^q)} \leq C\|e_h\|_{L^p(0,s+\delta_h;L^q)}^{1-1/p}\|e_h\|_{W^{1,p}(0,s+\delta_h;L^q)}^{1/p} \leq Ch^{k+1}(1-1/p) \leq Ch^{2-2/p}. \]

Using the inverse inequality again, we see
\[ \|e_h\|_{L^\infty(0,s+\delta_h;W^{1,\infty})} \leq Ch^{-1-N/q}\|e_h\|_{L^\infty(0,s+\delta_h;L^q)} \leq Ch^{2-2/p-N/q}, \]

Since \( p > 2 \) and \( q > N \), there exists \( h_2 > 0 \) such that when \( h < h_2 \)
\[ \|e_h\|_{L^\infty(0,s+\delta_h;W^{1,\infty})} < 1/2, \]

The mathematical induction on (44) is completed, which implies that (54)-(57) hold for any \( s \in (0,T] \). In particular, (54) implies (13).

Finally, we see that (14) is a simple consequence of (13). In fact, for any fixed \( q > N \), we have
\[ \|e_h\|_{L^\infty(0,T;L^q)} \leq C\|e_h\|_{L^p(0,T;L^q)}^{1-1/p}\|e_h\|_{W^{1,p}(0,T;L^q)}^{1/p} \leq C_p h^{k+1-(k+1)/p}, \]

where the Sobolev interpolation inequality and (56) have been used. In the above inequality, \( C_p \) is chosen as an increasing function of \( p \). Let \( h_0 = \min(h_1, h_2, C_3^{-3/(k+1)}) \).

Since the inequality above holds for any \( 2 < p < \infty \) and \( h < h_0 \), by choosing \( p \) such that \( C_p = h^{-(k+1)/p} \), we get
\[ \|e_h\|_{L^\infty(0,T;L^q)} \leq h^{k+1-\varepsilon}, \]

where
\[ \varepsilon = 2(k+1)/\gamma((k+1)\log(1/h)) \]
and \( \gamma(\zeta) \) is the inverse function of \( \zeta = p \log C_p \). Clearly,
\[ \lim_{h \to 0} \varepsilon = 0. \]

The proof of Theorem 1.2 is completed.

5. Conclusion

In this paper, we have presented a framework for optimal \( L^p \)-norm and almost optimal \( L^\infty \)-norm error estimates of finite element solutions for general nonlinear parabolic equations in polygons and polyhedra, with nonsmooth diffusion coefficients. This approach is based on the discrete maximal \( L^p \)-regularity of linear parabolic finite element equations. Most previous analyses on maximal \( L^p \) estimates were restricted to the problem in a smooth domain with stronger regularity assumptions on the diffusion coefficients.

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